ON THE DETERMINATION OF THE SHAPE OF A SUPERSONIC NOZZLE TAKING INTO CONSIDERATION THE VARIATION OF AIRCRAFT FLIGHT CONDITIONS

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The problem of designing the supersonic part of a fixed-area propelling nozzle of optimum shape for a certain trajectory, taking into account the variation of flight and engine operation conditions is considered below. The aircraft is assumed to be a material point of variable mass, and its drag at any instant is considered to be equal to the corresponding stationary value. The same approach is used in the analysis of flow in the nozzle. This implies that at any instant pressure and other parameters are defined (in a coordinate system of the nozzle) by equations of stationary flow in conditions prevailing at the nozzle inlet at the particular instant.

Besides the over-all results, a detailed investigation is made of two cases in which the use of the derived optimum conditions simplifies the solution of the problem. The first case occurs when throughout the flight the Mach number distribution at the nozzle inlet remains unchanged. It is found that in this case the optimum contour belongs to a family of contours corresponding to the solution of a variational problem with specified conditions. The second case is that of a plane and "short" nozzle with the flow at its inlet remaining throughout the whole flight uniform and supersonic. In this case the generatrix of the optimum nozzle is a straight line.

The problem of shaping the supersonic part of a plane or axisymmetric nozzle for maximum thrust at specified flow at its inlet and given external conditions has, to a great extent, been recently solved [1−4]. The formulation of the problem given in the following is due to the fact that in many applications the variation of flight conditions and of parameters at the inlet to the considered part of the nozzle is quite extensive, and that the flow at the inlet may vary not only because of changing flight conditions but, also, owing to the control of engine opertation. We note, incidentally, that the control of the operation mode is a problem of its own dealt with in numerous publications (see, e. g. [5−7]).

1. Let us consider the plane motion of an aircraft, assumed to be a material point, in the atmosphere and in the presence of a gravitational field. We assume that the vectors $\mathbf{V}$ of aircraft velocity and $\boldsymbol{\chi}$ of engine thrust coincide. Let t be the time, m the aircraft mass, h and l the vertical and horizontal coordinates, respectively, θ the angle of inclination of the trajectory to the horizon, G the mass fuel consumption per unit of time, and let $V = |\mathbf{V}|$, $\chi = |\boldsymbol{\chi}|$, and F_τ and F_n be the projections of vectors of external forces, respectively, on the tangent and normal to the trajectory. In this notation the aircraft motion is defined by equations

$$\begin{gathered}K_1 \equiv m^\cdot + G = 0, \qquad K_2 \equiv V^\cdot - (\chi - F_\tau) / m = 0 \\ K_3 \equiv h^\cdot - V \sin\theta = 0, \qquad K_4 \equiv \theta^\cdot + F_n / mV = 0 \\ K_5 \equiv l^\cdot - V\cos\theta = 0\end{gathered} \tag{1.1}$$

where the dot denotes a derivative with respect to t.

All parameters in (1.1) are assumed to be dimensionless. If h_*°, V_*° and m_*° are characteristic values of dimensions of length, velocity and mass, respectively, then h and l are normalized with respect to h_*°, velocity with respect to V_*°, time with respect to h_*° / V_*°, forces with respect to $m_*^\circ V_*^{\circ 2} / h_*^\circ$ and fuel consumption with respect to $m_*^\circ V_*^\circ / h_*^\circ$. For a plane aircraft all magnitudes here and in the following are assumed normalized with respect to its unit width.

Let us consider the flow in a plane ($\nu = 0$) or axisymmetric ($\nu = 1$) nozzle (Fig. 1) whose axis or plane of symmetry is assumed to be at every instant tangent to the aircraft trajectory. Let xy be a system of orthogonal coordinates attached to the nozzle. The flow at inlet to the investigated part of the nozzle ($x = 0$) is assumed to be at every instant uniform and supersonic (or sonic) and the gas nonviscous and not heat conducting. We confine our investigations to the case in which there are no shock waves in the region affected by the wall ab and bounded on the right by the characteristic fb of the first family. On these assumptions the flow in region $D = D\ (t)$ bounded by segments oa and od of the y- and x-axes, the characteristic db and the wall ab is at every instant defined by the equations

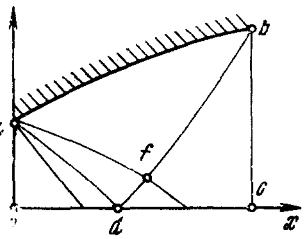

Fig. 1

$$L_1 \equiv u_y - v_x = 0,\ L_2 \equiv \rho u_x + \rho v_y + u\rho_x + v\rho_y + \nu\rho v y^{-1} = 0 \quad (1.2)$$

where ρ is the density, u and v are projections of the velocity vector on the x- and y-axes, and the subscripts x and y denote the related partial derivatives. Inertia forces produced by the acceleration of the aircraft have not been taken into consideration in (1.2).

All quantities in (1.2) including the gas pressure p are dimentionless. Independent variables are normalized with respect to the dimensional coordinate y_a° of point a, velocities with respect to w_0°, density with respect to ρ_0°, and pressure with respect to $\rho_0^\circ w_0^{\circ 2}$, where w_0° and ρ_0° are dimensional values of velocity and density at inlet to the considered part of the nozzle, i.e. $x = 0$ at a given instant. With this notation and on assumptions made with respect to conditions at $x = 0$, we have throughout the period of engine operation: $v \equiv 0,\ u \equiv \rho \equiv 1$, and the Mach number $M \geqslant 1$.

We further restrict ourselves to the case in which the equation of state and the expression of specific enthalpy together with the conditions for isentropic and constant-energy transformations, which in these circumstances are satisfied in D for any stationary flow, as is the case of a perfect gas, yield the equalities

$$\rho = \rho\ (w,\ p_0),\ p = p\ (w,\ p_0) \quad (1.3)$$

where w is the modulus of velocity, p_0 is the dimensionless pressure at $x = 0$, the functions appearing at the right are known, and $(\partial p/\partial w)_{p_0} = -\rho w$ and $(\partial\rho / \partial w)_{p_0} = -\rho w a^{-2}$, where a is the speed of sound.

Along the nozzle wall defined by equation $x = x\ (y)$ the no-leak condition

$$L \equiv x' - u / v = 0$$

is satisfied.

Here and in the following the prime denotes a total derivative with respect to y in the appropriate direction. In accordance with the normalization of (1.1) and (1.2) we have

$$\chi = \left(1 + p_0 + \int_a^b p y^\nu dy\right) k \qquad \left(k = \frac{2\pi^\nu (y_a{}^\circ)^{1+\nu} h_*{}^\circ \rho_0{}^\circ w_0{}^{\circ 2}}{m_*{}^\circ V_*{}^{\circ 2}}\right) \tag{1.4}$$

The dependence of p_0 and k on V, h and G is determined by the type of engine and its working process, and is assumed to be known. The form of (1.4) implies that the momentum at the air intake (in the case of an air-breathing jet engine) is included in the aircraft drag F_τ. The component F_τ also includes the base pressure acting on the end face $x \equiv x_b$ which may appear when the maximum admissible nozzle length is specified. We assume forces F_τ and F_n to be of the form

$$F_\tau = F^1 (V, h, x_b, y_b) + mg \sin\theta, \qquad F_n = F^2 (V, h) + mg \cos\theta ,$$

where F^1 and F^2 are known functions, and subscript b denotes, as before, parameters at the corresponding point. These expressions imply that the external outline of the aircraft belongs to a certain two-parameter family dependent on x_b and y_b.

2. Let us formulate the variational problem. We assume that at the initial instant $t = t_1$ the coordinates l_1, h_1, and the aircraft velocity V_1, angle θ_1 and mass m_1 are known. We shall denote the values of parameters at $t = t_i$ by subscript i. We have to determine the contour of a fixed-area propelling nozzle $x = x(y) \leqslant X$ passing through the given point a and providing for specified aircraft parameters at $t = t_1$ the maximum of a certain function Φ of parameters t_2, l_2, h_2, V_2, θ_2 and m_2 at the end point $t = t_2$ of the considered section of the trajectory. When $\Phi = -t_2$ we have the problem of time-optimal response, when $\Phi = -m_2$ that of minimum fuel consumption, when $\Phi = V_2$ that of maximum speed, etc. The remaining parameters, or part of these may in these problems be specified at $t = t_2$.

The aircraft motion and the pressure distribution along the nozzle wall, and consesently also χ, are determined in accordance with the equations and conditions of the preceding Section. Finally, $G = G(t)$ is considered to be either a known function, or as the control function to be determined simultaneously with the nozzle shape from the condition for optimum of functional Φ. In the latter case we assume that $G_{\min} \leqslant G(t) \leqslant G_{\max}$, where $G_{\min}$ and $G_{\max}$ are given constants. A flight with the engine shutdown is, also, possible when $G \equiv 0$ and $\chi = kp^+ y_b^{1+\nu} / (1+\nu)$, where p^+ is the so-called base pressure (dimensionless). The selection of $\rho_0{}^\circ w_0{}^{\circ 2}$ is made so as to have $p^+ \equiv$ const.

3. The necessary extremum conditions defining the optimum contour $\bar{ab}$, and if required, also, the control function $G(t)$, are derived from the analysis of the first variation of functional

$$I = \Phi + \int_{t_1}^{t_2} \left\{ \sum_{i=1}^{5} \lambda_i K_i + \int_a^b \beta L dy + \iint_D (\mu_1 L_1 + \mu_2 L_2)\, dx dy \right\} dt \tag{3.1}$$

where $\lambda_i(t)$, $\beta(y, t)$ and $\mu_i(x, y, t)$ are Lagrangian multipliers. Within admissible variations (limits) the variations of functionals I and Φ coincide.

Applying to (3.1) the usual method of variation and of selection of Lagrangian multipliers, we find that the optimum contour is defined by conditions

$$\int_{t_1}^{t_2} y^\nu \rho v \left(Wu - \frac{\mu_2}{y^\nu}\right) dt = 0 , \qquad \text{along } ab \tag{3.2}$$

$$\int_{t_1}^{t_2} \frac{\lambda_2}{m}(F_{x_b}^1 + ky^\nu \rho v^2 \operatorname{tg}\alpha)_b dt \geqslant 0 \qquad \text{(cont.)}$$

$$\int_{t_1}^{t_2} \frac{\lambda_2}{m}\{F_{y_b}^1 + ky^\nu (\rho uv \operatorname{tg}\alpha - p)\}_b dt = 0$$

where the second and third conditions determine x_b and y_b, respectively, and when $x_b < X$, the condition for x_b reduces to an equality. At $t > t_3$, where t_3 is defined below, we have in the integrands $u = v = 0$ and $p = p^+$.

The subscripts x_b, y_b, V, h, etc. denote in (3.2) and in the following the corresponding partial derivatives; $\alpha = \arcsin(a / w)$ is the Mach angle, and $W = \lambda_2 k / m$.

Multipliers λ_i including multiplier λ_2 are determined by the system

$$m^2 V\lambda_1^\cdot - \lambda_2 V(\chi - F_\tau) - \lambda_2 V mg \sin\theta + \lambda_4 F^2 = 0$$

$$mV^2\lambda_2^\cdot + \lambda_2 V^2 \{\chi (\ln k)_V + k\ (p_0)_V - F_V{}^1\} + \varepsilon m V^2 (p_0)_V +$$

$$+ \lambda_3 m V^2 \sin\theta + \lambda_4 (F_n - VF_V{}^2) + \lambda_5 m V^2 \cos\theta = 0 \qquad (3.3)$$

$$mV\lambda_3^\cdot + \lambda_2 V\{\chi (\ln k)_h + k(p_0)_h - F_h{}^1\} + \varepsilon m V(p_0)_h - \lambda_4 F_h{}^2 = 0$$

$$V\lambda_4^\cdot - \lambda_2 Vg \cos\theta + \lambda_3 V^2 \cos\theta + \lambda_4 g \sin\theta - \lambda_5 V^2 \sin\theta = 0$$

$$\lambda_5^\cdot = 0 \quad \left(\varepsilon = k\frac{\lambda_2}{m}\int_a^b \left(\frac{\partial p}{\partial p_0}\right)_w y^\nu dy + \iint_D \rho\mu_2 (uR_x + vR_y)\, dxdy\right)$$

where $R \equiv (\partial \ln \rho / \partial p_0)_w$ and $(\partial p / \partial p_0)_w$ are calculated in conformity with (1.3).

If any of the parameters m_2, V_2, ... is free or is being optimized, the boundary condition for the multiplier λ_i introducing the related equation from (1.1) is of the form

$$\lambda_1 = -\Phi_{m_2}, \quad \lambda_2 = -\Phi_{V_2}, \quad \lambda_3 = -\Phi_{h_2}, \quad \lambda_4 = -\Phi_\theta, \qquad (3.4)$$
$$\lambda_5 = -\Phi_{l_2} \qquad (\text{at} \quad t = t_2)$$

and the value of t_2 is either given or is determined by condition

$$\{mV\Phi_{t_2} + \lambda_1 mVG - \lambda_2 V(\chi - F_\tau) - \lambda_3 mV^2 \sin\theta + \lambda_4 F_n - \lambda_5 mV^2 \cos\theta\}_2 = 0 \quad (3.5)$$

If any of the parameters at the end point of the trajectory is specified, the related multiplier λ_i at $t = t_2$ is not derived from (3.4) but selected so as to obtain a contour which would satisfy the particular requirement.

It may happen that the specified $m = m_2$ is achieved at $t_3 < t_2$. We then have for $t_3 < t < t_2$ a flight with the engine shutdown. In such case the first equation of system (3.3) is integrable for $t \leqslant t_3$ only, and in (1.1) and (3.3) for $t > t_3$ with G and λ_1. omitted in (3.5). The boundary condition for λ_1 is then of the form

$$\lambda_1 mG - \lambda_2 [\chi] = 0 \qquad \text{for } t \to t_3$$

where $[\chi]$ is the sudden change of χ at the instant of engine shutdown. Finally, if at a certain instant $t = t_j$ function $G(t)$ becomes discontinuous, the multipliers λ_i remain continuous at the point of discontinuity, except in the case of $t_j = t_3$ considered above in which λ_1 is defined at $t \leqslant t_3$ only.

Multipliers μ_1 and μ_2 in the subregions of their discontinuity in D satisfy equations

$$a^2\mu_{1y} + \rho(a^2 - u^2)\mu_{2x} - \rho uv\mu_{2y} + \nu\rho uvy^{-1}\mu_2 = 0 \qquad (3.6)$$

$$a^2\mu_{1x} + \rho uv\mu_{2x} + \rho(v^2 - a^2)\mu_{2y} + \nu\rho(a^2 - v^2)y^{-1}\mu_2 = 0$$

which have the same characteristics as the equations of flow. Along the characteristics the following relationships are satisfied:

$$\mu_1' \operatorname{tg} \alpha \mp \mu_2' \rho \pm \nu \rho y^{-1} \mu_2 = 0 \tag{3.7}$$

Here and in the following the upper sign corresponds to characteristics of the first family and the lower to those of the second.

Generally, the regions of continuity of μ_i are separated by lines of discontinuity which can be characteristics of the first or second family [4, 8, 9]. If $[\mu_i]$ represents the difference of μ_i up- and downstream of the discontinuity (in the direction of the flow of gas), then the following relationship

$$[\mu_1] \pm [\mu_2] \rho \operatorname{ctg} \alpha = 0 \tag{3.8}$$

is satisfied at the discontinuities.

The boundary conditions which together with (3.8) are required for integrating (3.6), are specified at the nozzle axis, along the contour ab, and along the characteristic db, and are of the form

$$\mu_1 = 0 \ \text{ for } \ y = 0, \qquad \mu_1 = y^\nu \rho v W \ \text{ along } ab, \tag{3.9}$$
$$\mu_1 + \mu_2 \rho \operatorname{ctg} \alpha = 0 \quad \text{along } db$$

Conditions (3.9) together with related equation of (3.7) make possible the determination of μ_i along db. As the result we obtain that along db

$$\mu_1 = \sqrt{C y^\nu \rho \operatorname{ctg} \alpha}, \qquad \mu_2 = -\sqrt{C y^\nu \operatorname{tg} \alpha / \rho} \tag{3.10}$$
$$C = (y^\nu \rho v^2 \operatorname{tg} \alpha)_b \ W^2$$

Here W is the same as in (3.2), and the signs of roots are chosen in accordance with condition (3.9).

In the case considered here the discontinuity of μ_i occurs along the characteristic ad, where

$$[\mu_1] = -\sqrt{C y^\nu \rho \operatorname{ctg} \alpha}, \qquad [\mu_2] = \sqrt{C y^\nu \operatorname{tg} \alpha / \rho} \tag{3.11}$$

with constant C and the signs of square roots being the same as in (3.10).

In the plane case $\nu = 0$ a discontinuity along ad and equalities (3.11) are unavoidable, as seen from the juxtaposition of conditions (3.9) along the axis and of the first of formulas (3.10) with subsequent use of (3.8). It follows from (3.10) that in the axisymmetric case ($\nu = 1$) the multiplier μ_1 along db tends to zero with decreasing distance from the axis. The discontinuity of μ_i along ad is caused by the discontinuity of μ_{1x} at point d. Formulas (3.11) can be derived either as in [9], or by passing to limit in the examination of a nozzle with a cylindrical central body of radius r. Equations (3.9) and (3.10) remain valid for such a nozzle, if $y = r$ is substituted for $y = 0$ and $d'b$ for db, where d' is the point of intersection of characteristic $d'b$ of the first family with the central body. Hence along ad' the equalities (3.11) remain valid for any $r > 0$. For sufficiently small r characteristics ad' and ad virtually coincide, and the intensity of discontinuity of μ_i, defined by formulas (3.11), becomes independent of r at any arbitrary fixed y. This proves the validity of transition to limit at $r \to 0$.

Let us assume that contour ab has been chosen for certain reasons. The simultaneous solution of Eqs. (1.1) and (1.2) together with related initial and boundary conditions makes it possible to determine the variation of aircraft parameters in terms of t, as well as the distribution of flow parameters in the nozzle at any instant. This and the use of equations and conditions (3.3)–(3.11) permit, in turn, to determine $\lambda_i\ (t)$ and $\mu_i\ (x,\ y,\ t)$, including λ_2 and μ_2 which appear in the optimization condition (3.2). The verification of these conditions can be used as the basis of the procedure for designing the optimum nozzle.

Any region which includes D as a subregion may be substituted in (3.1) for D. In particular, region D' bounded by the contour $oabco$, which is independent of t, can be taken for such region. In this case the conditions along db are replaced by equalities $\mu_1 \equiv \mu_2 \equiv 0$ along cb from which, by virtue of (3.6) and (3.9), follows that $\mu_i \equiv 0$ in the triangle dbc, and that db is the line of discontinuity of μ_i. As the result, for the determination of μ_i in D we obtain along db the previous boundary conditions (3.10). There are of course circumstances in which shock waves are generated in the triangle dbc, in spite of fulfilment of the condition for vortex-free flow in D, which by itself substantially restricts the domain of applicability of this method of analysis. Nevertheless, even in this case, a rigorous analysis with the substitution in dbc of equations of rotational flow for (1.2), and taking into consideration the relationships obtaining at shock waves, yields for μ_i in region D and at its boundaries the previously obtained results.

Completing the analysis of the necessary conditions for maximum Φ, we note that in optimizing the fuel consumption control function $G\ (t)$ along with the shape of the nozzle, the former is selected on the basis of the condition for nonpositivity of the first variation

$$\delta\Phi = \int_{t_1}^{t_2} Q\delta G dt + \sum_j S_j \Delta t_j$$

Here $t = t_j$ are the instants of resetting of $G\ (t)$, and Q and S are defined by formulas

$$Q = \lambda_1 - \frac{\lambda_2}{m} \{\chi\,(\ln k)_G + k\ (p_0)_G\} - \varepsilon (p_0)_G, \qquad S = \lambda_1\,[G] - \frac{\lambda_2}{m}\,[\chi]$$

where $[G]$ and $[\chi]$ are sudden changes of G and χ due to resetting of controls.

It should be borne in mind when using this method that in accordance with (1.2) the flow in the nozzle is considered in a quasi-stationary approximation. Hence, if a specified or obtained by optimization $G\ (t)$ has points of discontinuity, the contour designed in conformity with conditions (3.2), will be close to the optimum (for a real nonstationary flow), only when the total relative time of fast transitory processes in the nozzle is short (*). Any so-called sliding optimum modes must be excluded for the same reason from consideration, when using this approach [10].

4. Let us assume that for any possible variation of dimensional parameters at the nozzle inlet the dimensionless pressure p_0 or the Mach number (which for the gas considered here is equivalent) are constant. This occurs, e. g. when the initial section of the nozzle coincides with its throat. We note, incidentally, that, when in this case at $x = 0$ the distribution of dimensionless velocities u and v is also invariant, the assumption of a uniform flow at the inlet section is not obligatory.

Under such conditions the fields of dimensionless flow parameters in the nozzle, the characteristic net, and, in particular, the closing characteristic db are the same at all

*) The condition of validity of the quasi-stationary approximation must be, strictly speaking, formulated as a limitation imposed on $|\,\dot{G}\,|$.

points of the trajectory. Nevertheless multipliers μ_i appearing in condition (3.9) at the contour depend on t via function $W(t)$.

Let us denote by a small circle superscript (not to be confused with dimensional parameters) parameter values averaged over the time t of engine operation, i.e. over the period $t_3 - t_1$. We then integrate the equations and the boundary conditions for μ_i with respect to t, taking into consideration that the flow parameters appearing in these equations and conditions are independent of t. As the result, we find that μ_i° satisfy the same equations and conditions as μ_i, if in (3.9) and (3.10) constant

$$W^\circ = \frac{1}{t_3 - t_1} \int_{t_1}^{t_3} W(t)\, dt$$

is substituted for function $W(t)$.

The first of the optimixation conditions (3.2) can now be written in the form of equation

$$(W^\circ u - y^{-\nu} \mu_2^\circ)' = 0 \text{ along } ab$$

which after transformation becomes

$$\mu_2^\circ = y^\nu (u + C_1)\, W^\circ \text{ along } ab \tag{4.1}$$

where C_1 is the constant of integration.

It can be also shown that, as in [11], (4.1) and the second of equalities in (3.9) written for μ_1° are the integrals of equations for μ_i° and, as a corollary, that the optimum contour ab belongs to a known family of optimum contours ensuring the maximum thrust for specified conditions at the inlet to the nozzle supersonic part and of back pressure, and maximum permitted [nozzle] length [1-4]. In accordance with the condition in (3.9) related to db and with the derived above integrals for μ_i° which yield the solution of the problem of Lagrangian multipliers averaged over t in the triangle abf, the parameters along segment fb of characteristic ab of this family of nozzles satisfy the equality

$$y^\nu \rho v^2 tg\, \alpha = \text{const} \quad \text{along } fb \tag{4.2}$$

It can be shown, moreover, that

$$u + v \operatorname{tg} \alpha + C_1 = 0 \text{ along } fb \tag{4.3}$$

where C_1 is the same constant as in (4.1).

By virtue of above considerations the optimum contour must be selected in this case from the family of contours for which equalities (4.2) and (4.3) hold, and the integral in the expression (1.4) for χ is a function of x_b and y_b only. It can be shown that

$$\chi_{x_b} = k(y^\nu \rho v^2 \operatorname{tg} \alpha)_b, \qquad \chi_{y_b} = k y_b^\nu (p - \rho u v \operatorname{tg} \alpha)_b$$

hence the second and third of conditions (3.2) defining x_b and y_b are rewritten as

$$\int_{t_1}^{t_2} \frac{\lambda_2}{m} (F^1_{x_b} - \chi_{x_b})\, dt \geqslant 0, \qquad \int_{t_1}^{t_2} \frac{\lambda_2}{m} (F^1_{y_b} - \chi_{y_b})\, dt = 0 \tag{4.4}$$

All of the previous statements relative to formulas (3.2) and, in particular, that about flight with the engine shut down ($t > t_3$), remain valid.

When a horizontal flight at constant velocity with the thrust equal to the drag and the lift balanced by gravity, and $G(t) \equiv \text{const}$, the expressions in parentheses in (4.4) are independent of t and can be taken out of the integrand. Under such conditions δV is also independent of t and, consequently, λ_2 must be determined not by (3.3) and (3.4) but by condition

$$\int_{t_1}^{t_2}\left\{\frac{\lambda_2}{m}(\chi_V - F_V{}^1) + \lambda_5\right\}dt - \Phi_{V_?} = 0 \qquad (4.5)$$

Let us now consider the problem of maximum flight range $(\Phi = l_2)$. With the use of (3.3), (3.4) and (3.5) it can be shown that

$$(\chi_V - F_V{}^1)\int_{t_1}^{t_2}\frac{\lambda_2}{m}\,dt = t_2 - t_1 > 0$$

If in this case the mode is stable, the expression in parentheses is negative, and conditions (4.4) become

$$k\,(y^\nu \rho v^2 \operatorname{tg}\alpha)_b - F_{xb}{}^1 \geqslant 0, \quad k y_b{}^\nu\,(\rho u v \operatorname{tg}\alpha - p)_b - F_{yb}{}^1 = 0$$

5. Another case in which the design of a nozzle of optimum contour is substantially simplified is that when for $t_1 < t < t_3$ the flow at the nozzle inlet is supersonic and uniform, the nozzle is plane $(\nu = 0)$, and its maximum admissible length X such that the pattern shown in Fig. 2a prevails at every instant. A particular feature of this pattern is that the characteristic db, although varying in time, always intersects the whole beam of rarefaction waves generated in the flow by the kink at point a, and, also, its distortion due to variation of the Mach number at $x = 0$. The characteristics of the second family in this beam to the left of db are rectilinear, and the parameters along each of these characteristics (in particular along af) are constant.

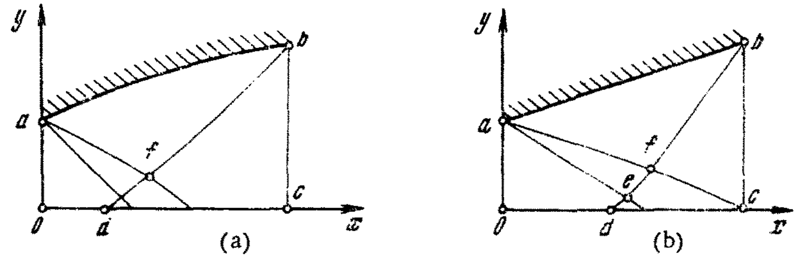

Fig. 2

It will be readily seen that for $\nu = 0$ functions u, v, ρ, μ_1 and μ_2, dependent only on t but independent of x and y, satisfy the equations of flow (1.2) as well as Eqs. (3.6) for μ_i. A flow uniform with respect to x and y in the triangle abf satisfies also the condition of constancy of flow parameters realized in this case along af. The multipliers μ_1 and μ_2 in abf may be expressed in terms of gasdynamic parameters by using the conditions along ab and fb of (3.9). Since u and μ_2 are independent of x and y, it is readily seen that

$$(Wu - \mu_2)' = 0 \quad \text{along } ab$$

Hence the first of the optimization conditions (3.2) is not only completely satisfied with respect to t but, also, at every instant of time. In this case the optimum contour is to be selected from a family of contours with a straight-line generatrix (Fig. 2b). The coordinates of the end point b of the contour ab must satisfy the second and third of conditions (3.2) which may be here presented in the form (4.4). Owing to the constancy of flow parameters in triangle abf and to the left of the first of the characteristics of the beam, region D in the expression for ε in (3.3) reduces to triangle afe. Multipliers μ_i in the beam to the left of ef are constant along every characteristic of the second family, and are defined by the values of parameters along ef found from formula (3.10) for $\nu = 0$. We note, incidentally, that in this case

$$\varepsilon = (y_b - y_a)\,(dp_b / dp_0)$$

where (dp_b / dp_0) is determined by formulas for flows past an obtuse angle.

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